3 CHAPTER

Electric Flux Density, Gauss's Law, and Divergence

After drawing a few of the fields described in the previous chapter and becoming familiar with the concept of the streamlines that show the direction of the force on a test charge at every point, it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing "something" is present.

It is very helpful to invent an *electric flux* that streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

This chapter introduces and uses the concept of electric flux and electric flux density to again solve several of the problems presented in Chapter 2. The work here turns out to be much easier, and this is due to the extremely symmetrical problems that we are solving. ■

3.1 ELECTRIC FLUX DENSITY

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now-famous work on induced electromotive force, which we will discuss in Chapter 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply *dielectric*) that would occupy the entire volume between the concentric spheres. We will immediately use his findings about dielectric materials, for we are restricting our attention to fields in free space until Chapter 6. At that time we will see that the materials he used will be classified as ideal dielectrics.

His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux,* or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in coulombs.

We can obtain more quantitative information by considering an inner sphere of radius a and an outer sphere of radius b, with charges of Q and $-Q$, respectively (Figure 3.1). The paths of electric flux Ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge $Q(= \Psi)$ Cs distributed uniformly over a surface having an area of $4\pi a^2$ m^2. The density of the flux at this surface is $\Psi/4\pi a^2$ or $Q/4\pi a^2$ C/m^2, and this is an important new quantity.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we will use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric

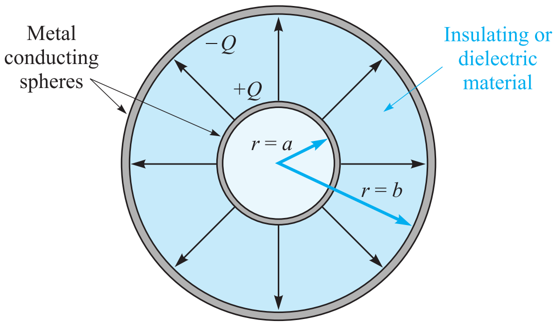

Figure 3.1 The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of **D** are not functions of the dielectric between the spheres.

field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Referring again to Figure 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

If we now let the inner sphere become smaller and smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \tag{1}$$

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

This result should be compared with Section 2.2, Eq. (9), the radial electric field intensity of a point charge in free space,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

In free space, therefore,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \quad \text{(free space only)} \tag{2}$$

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space,

$$\boxed{\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\mathbf{a}_R} \quad \text{(free space only)} \tag{3}$$

where this relationship was developed from the field of a single point charge. In a similar manner, (1) leads to

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R} \tag{4}$$

and (2) is therefore true for any free-space charge configuration; we will consider (2) as defining **D** in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between **D** and **E** will be slightly more complicated than (2).

Because **D** is directly proportional to **E** in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for a few reasons. First, **D** is associated with the flux concept, which is an important new idea. Second, the **D** fields we obtain will be a little simpler than the corresponding **E** fields, because ϵ_0 does not appear.

D3.1. Given a 60-µC point charge located at the origin, find the total electric flux passing through: (*a*) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (*b*) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (*c*) the plane $z = 26$ cm.

Ans. 7.5 µC; 60 µC; 30 µC

D3.2. Calculate **D** in rectangular coordinates at point $P(2, -3, 6)$ produced by: (*a*) a point charge $Q_A = 55$ mC at $Q(-2, 3, -6)$; (*b*) a uniform line charge $\rho_{LB} = 20$ mC/m on the x axis; (*c*) a uniform surface charge density $\rho_{SC} = 120\,\mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z\ \mu\text{C/m}^2$; $-212\mathbf{a}_y + 424\mathbf{a}_z\ \mu\text{C/m}^2$; $60\mathbf{a}_z\ \mu\text{C/m}^2$

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, because one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but $+Q$ coulombs on any inner conductor would produce an induced charge of $-Q$ coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce $-Q$ coulombs on the tin can.[1]

These generalizations of Faraday's experiment lead to the following statement, which is known as *Gauss's law*:

> *The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.*

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have, but in providing a mathematical form for this statement, which we will now obtain.

Let us imagine a distribution of charge, shown as a cloud of point charges in Figure 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector $\mathbf{D}$ will have some value $\mathbf{D}_S$, where the subscript S merely reminds us that $\mathbf{D}$ must be evaluated at the surface, and $\mathbf{D}_S$ will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction that may be associated with $\Delta\mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two such normals, and the ambiguity is removed by specifying the outward normal whenever the surface is closed and "outward" has a specific meaning.

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[1] If it were a perfect insulator, the soup could even be left in the can without any difference in the results.

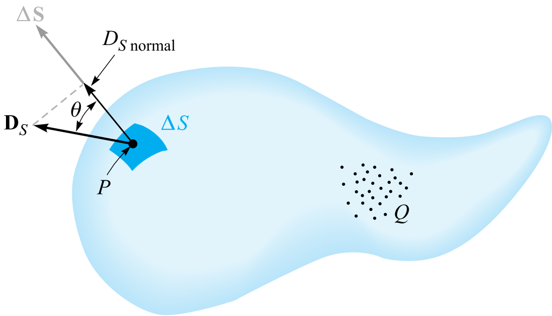

Figure 3.2 The electric flux density $\mathbf{D}_S$ at P arising from charge Q. The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta \mathbf{S}$.

At any point P, consider an incremental element of surface ΔS and let $\mathbf{D}_S$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure 3.2. The flux crossing ΔS is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta \, \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chapter 1.

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\\text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\,d\phi\;d\rho$, or $r^2 \sin\theta\,d\theta\,d\phi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an S below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q \tag{5}$$

The charge enclosed might be several point charges, in which case

$$Q = \Sigma Qn$$

or a line charge,

$$Q = \int \rho_L \, dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv \tag{6}$$

a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

EXAMPLE 3.1

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a.

Solution. We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

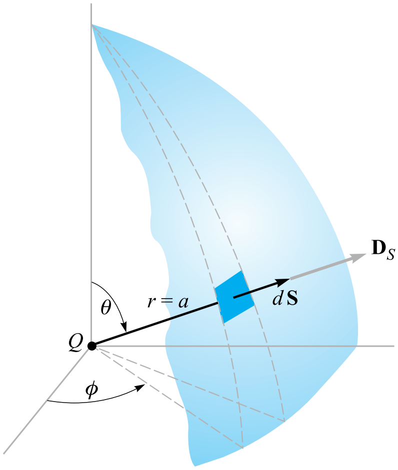

Figure 3.3 Applying Gauss's law to the field of a point charge Q on a spherical closed surface of radius a. The electric flux density $\mathbf{D}$ is everywhere normal to the spherical surface and has a constant magnitude at every point on it.

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.[2] Integrating gives

$$\int_0^{2\pi} \frac{Q}{4\pi}\left(-\cos\theta\right)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

WW Animations

and we obtain a result showing that Q coulombs of electric flux are crossing the surface, as we should since the enclosed charge is Q coulombs.

D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m^2 in free space: (a) find $\mathbf{E}$ at point $P(r = 2, \theta = 25^\circ, \phi = 90^\circ)$; ($b$) find the total charge within the sphere $r = 3$; (c) find the total electric flux leaving the sphere $r = 4$.

Ans. $135.5\mathbf{a}_r$ V/m; 305 nC; 965 nC

D3.4. Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (a) two point charges, 0.1 µC at $(1, -2, 3)$ and $\frac{1}{7}$ µC at $(-1, 2, -2)$; (b) a uniform line charge of π µC/m at $x = -2, y = 3$; (c) a uniform surface charge of 0.1 µC/m^2 on the plane $y = 3x$.

Ans. 0.243 µC; 31.4 µC; 10.54 µC

[2] Note that if θ and ϕ both cover the range from 0 to 2π, the spherical surface is covered twice.

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

We now consider how we may use Gauss's law,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

to determine $\mathbf{D}_S$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.

The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

This allows us to replace the dot product with the product of the scalars D_S and dS and then to bring D_S outside the integral sign. The remaining integral is then $\int_S dS$ over that portion of the closed surface which $\mathbf{D}_S$ crosses normally, and this is simply the area of this section of that surface. Only a knowledge of the symmetry of the problem enables us to choose such a closed surface.

Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements previously listed. The surface in question is obviously a spherical surface, centered at the origin and of any radius r. $\mathbf{D}_S$ is everywhere normal to the surface; D_S has the same value at all points on the surface.

Then we have, in order,

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{\text{sph}} D_S dS$$

$$= D_S \oint_{\text{sph}} dS = D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi r^2 D_S$$

and hence

$$D_S = \frac{Q}{4\pi r^2}$$

Because r may have any value and because $\mathbf{D}_S$ is directed radially outward,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

which agrees with the results of Chapter 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which we may apply to other problems, including several to which Coulomb's law is almost incapable of supplying an answer.

Are there any other surfaces which would have satisfied our two conditions? The student should determine that such simple surfaces as a cube or a cylinder do not meet the requirements.

As a second example, let us reconsider the uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$. We must first know the symmetry of the field, and we may consider this knowledge complete when the answers to these two questions are known:

1. With which coodinates does the field vary (or of what variables is D a function)?
2. Which components of $\mathbf{D}$ are present?

In using Gauss's law, it is not a question of using symmetry to simplify the solution, for the application of Gauss's law depends on symmetry, and *if we cannot show that symmetry exists then we cannot use Gauss's law* to obtain a solution. The preceding two questions now become "musts."

From our previous discussion of the uniform line charge, it is evident that only the radial component of $\mathbf{D}$ is present, or

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

and this component is a function of ρ only.

$$D_\rho = f(\rho)$$

The choice of a closed surface is now simple, for a cylindrical surface is the only surface to which D_ρ is everywhere normal, and it may be closed by plane surfaces normal to the z axis. A closed right circular cylinder of radius ρ extending from $z = 0$ to $z = L$ is shown in Figure 3.4.

We apply Gauss's law,

$$Q = \oint_{\text{cyl}} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{\text{sides}} dS + 0 \int_{\text{top}} dS + 0 \int_{\text{bottom}} dS$$
$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L$$

and obtain

$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is

$$Q = \rho_L L$$

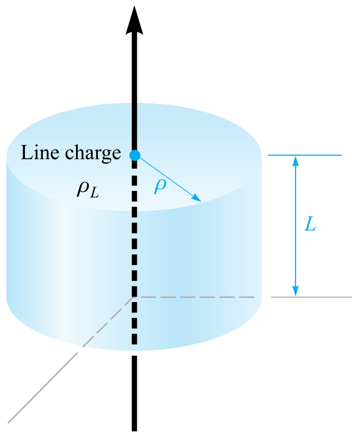

Figure 3.4 The gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ. **D** is constant in magnitude and everywhere perpendicular to the cylindrical surface; **D** is parallel to the end faces.

giving

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

or

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Comparing with Section 2.4, Eq. (16), shows that the correct result has been obtained and with much less work. Once the appropriate surface has been chosen, the integration usually amounts only to writing down the area of the surface at which **D** is normal.

The problem of a coaxial cable is almost identical with that of the line charge and is an example that is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent (Figure 3.5). We will assume a charge distribution of ρ_S on the outer surface of the inner conductor.

Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ. A right circular cylinder of length L and radius ρ, where $a < \rho < b$, is necessarily chosen as the gaussian surface, and we quickly have

$$Q = D_S 2\pi\rho L$$

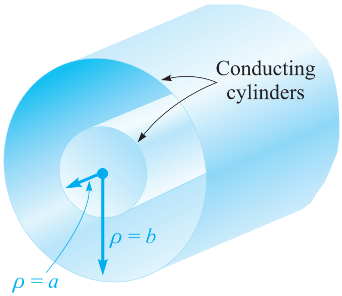

Figure 3.5 The two coaxial cylindrical conductors forming a coaxial cable provide an electric flux density within the cylinders, given by $D_\rho = a\rho_S/\rho$.

The total charge on a length L of the inner conductor is

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_S a \, d\phi \, dz = 2\pi a L \rho_S$$

from which we have

$$D_S = \frac{a\rho_S}{\rho} \qquad \mathbf{D} = \frac{a\rho_S}{\rho}\mathbf{a}_\rho \qquad (a < \rho < b)$$

This result might be expressed in terms of charge per unit length because the inner conductor has $2\pi a\rho_S$ coulombs on a meter length, and hence, letting $\rho_L = 2\pi a \rho_S$,

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

Illustrations

and the solution has a form identical with that of the infinite line charge.

Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

or

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

What would happen if we should use a cylinder of radius ρ, $\rho > b$, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

$$0 = D_S 2\pi\rho L \qquad (\rho > b)$$

$$D_S = 0 \qquad (\rho > b)$$

An identical result would be obtained for $\rho < a$. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a "shield"), and there is no field within the center conductor.

Our result is also useful for a *finite* length of coaxial cable, open at both ends, provided the length L is many times greater than the radius b so that the nonsymmetrical conditions at the two ends do not appreciably affect the solution. Such a device is also termed a *coaxial capacitor*. Both the coaxial cable and the coaxial capacitor will appear frequently in the work that follows.

EXAMPLE 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55\ \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outer cyl}} = \frac{Q_{\text{outer cyl}}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39\ \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho}\ \text{nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{1079}{\rho}\ \text{V/m}$$

Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

D3.5. A point charge of 0.25 μC is located at $r = 0$, and uniform surface charge densities are located as follows: 2 mC/m^2 at $r = 1$ cm, and -0.6 mC/m^2 at $r = 1.8$ cm. Calculate **D** at: (*a*) $r = 0.5$ cm; (*b*) $r = 1.5$ cm; (*c*) $r = 2.5$ cm. (*d*) What uniform surface charge density should be established at $r = 3$ cm to cause $\mathbf{D} = 0$ at $r = 3.5$ cm?

Ans. $796\mathbf{a}_r$ μC/m^2; $977\mathbf{a}_r$ μC/m^2; $40.8\mathbf{a}_r$ μC/m^2; -28.3 μC/m^2

3.4 APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

We are now going to apply the methods of Gauss's law to a slightly different type of problem—one that does not possess any symmetry at all. At first glance, it might seem that our case is hopeless, for without symmetry, a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties and that is to choose such a very small closed surface that **D** is *almost* constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we will not obtain the value of **D** as our answer but will instead receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

Let us consider any point P, shown in Figure 3.6, located by a rectangular coordinate system. The value of **D** at the point P may be expressed in rectangular components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, **D** is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned} \int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \, \Delta z \, \mathbf{a}_x \\ &\doteq D_{\text{x,front}} \Delta y \, \Delta z \end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned} D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \end{aligned}$$

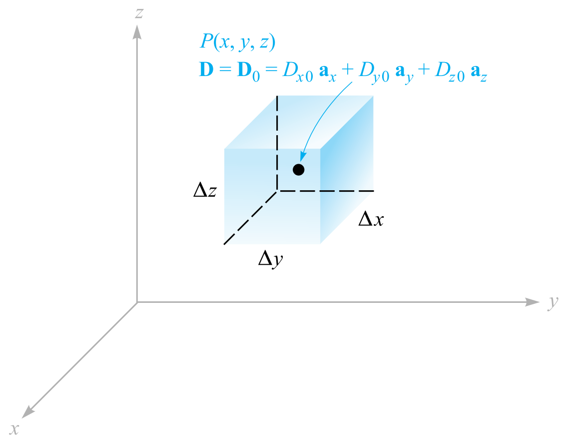

Figure 3.6 A differential-sized gaussian surface about the point P is used to investigate the space rate of change of **D** in the neighborhood of P.

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned}\int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \, \Delta z\end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v \tag{7}$$

The expression is an approximation which becomes better as Δv becomes smaller, and in the following section we shall let the volume Δv approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv and have as a result the approximation (7) stating that

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v \tag{8}$$

EXAMPLE 3.3

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y \, \mathbf{a}_x - e^{-x} \cos y \, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$
$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$
$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta v$. If Δv is 10^{-9} m^3, then we have enclosed about 2 nC.

D3.6. In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z$ pC/m^2. (*a*) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2$, $1 < y < 3$, in the $\mathbf{a}_z$ direction. (*b*) Find $\mathbf{E}$ at $P(2, -1, 3)$. (*c*) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m^3.

Ans. 1365 pC; $-146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; -2.38×10^{-21} C

3.5 DIVERGENCE AND MAXWELL'S FIRST EQUATION

We will now obtain an exact relationship from (7), by allowing the volume element Δv to shrink to zero. We write this equation as

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \tag{9}$$

in which the charge density, ρ_v, is identified in the second equality.

The methods of the previous section could have been used on any vector $\mathbf{A}$ to find $\oint_S \mathbf{A} \cdot d\mathbf{S}$ for a small closed surface, leading to

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{10}$$

where $\mathbf{A}$ could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, *divergence*. The divergence of $\mathbf{A}$ is defined as

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$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{11}$$

and is usually abbreviated div $\mathbf{A}$. The physical interpretation of the divergence of a vector is obtained by describing carefully the operations implied by the right-hand side of (11), where we shall consider $\mathbf{A}$ to be a member of the flux-density family of vectors in order to aid the physical interpretation.

The divergence of the vector flux density **A** *is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

The physical interpretation of divergence afforded by this statement is often useful in obtaining qualitative information about the divergence of a vector field without resorting to a mathematical investigation. For instance, let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through *any* closed surface lying entirely within the water must be zero, for water is essentially incompressible, and the water entering and leaving different regions of the closed surface must be equal. Hence the divergence of this velocity is zero.

If, however, we consider the velocity of the air in a tire that has just been punctured by a nail, we realize that the air is expanding as the pressure drops, and that consequently there is a net outflow from any closed surface lying within the tire. The divergence of this velocity is therefore greater than zero.

A positive divergence for any vector quantity indicates a *source* of that vector quantity at that point. Similarly, a negative divergence indicates a *sink*. Because the divergence of the water velocity above is zero, no source or sink exists.[3] The expanding air, however, produces a positive divergence of the velocity, and each interior point may be considered a source.

Writing (9) with our new term, we have

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \qquad \text{(rectangular)} \tag{12}$$

This expression is again of a form that does not involve the charge density. It is the result of applying the definition of divergence (11) to a differential volume element in *rectangular coordinates*.

If a differential volume unit $\rho\, d\rho\, d\phi\, dz$ in cylindrical coordinates, or $r^2 \sin\theta\, dr\, d\theta\, d\phi$ in spherical coordinates, had been chosen, expressions for divergence involving the components of the vector in the particular coordinate system and involving partial derivatives with respect to the variables of that system would have been obtained. These expressions are obtained in Appendix A and are given here for convenience:

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)} \tag{13}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \qquad \text{(spherical)} \tag{14}$$

These relationships are also shown inside the back cover for easy reference.

It should be noted that the divergence is an operation which is performed on a vector, but that the result is a scalar. We should recall that, in a somewhat similar way, the dot or scalar product was a multiplication of two vectors which yielded a scalar.

For some reason, it is a common mistake on meeting divergence for the first time to impart a vector quality to the operation by scattering unit vectors around in

[3] Having chosen a differential element of volume within the water, the gradual decrease in water level with time will eventually cause the volume element to lie above the surface of the water. At the instant the surface of the water intersects the volume element, the divergence is positive and the small volume is a source. This complication is avoided above by specifying an integral point.

the partial derivatives. Divergence merely tells us *how much* flux is leaving a small volume on a per-unit-volume basis; no direction is associated with it.

We can illustrate the concept of divergence by continuing with the example at the end of Section 3.4.

EXAMPLE 3.4

Find div **D** at the origin if $\mathbf{D} = e^{-x} \sin y\, \mathbf{a}_x - e^{-x} \cos y\, \mathbf{a}_y + 2z\mathbf{a}_z$.

Solution. We use (10) to obtain

$$\text{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$
$$= -e^{-x} \sin\, y + e^{-x} \sin\, y + 2 = 2$$

The value is the constant 2, regardless of location.

If the units of **D** are C/m^2, then the units of div **D** are C/m^3. This is a volume charge density, a concept discussed in the next section.

D3.7. In each of the following parts, find a numerical value for div **D** at the point specified: (*a*) $\mathbf{D} = (2xyz - y^2)\mathbf{a}_x + (x^2z - 2xy)\mathbf{a}_y + x^2y\mathbf{a}_z$ C/m^2 at $P_A(2, 3, -1)$; (*b*) $\mathbf{D} = 2\rho z^2 \sin^2\phi\, \mathbf{a}_\rho + \rho z^2 \sin 2\phi\, \mathbf{a}_\phi + 2\rho^2 z \sin^2\phi\, \mathbf{a}_z$ C/m^2 at $P_B(\rho = 2, \phi = 110^\circ, z = -1)$; (*c*) $\mathbf{D} = 2r \sin\theta \cos\phi\, \mathbf{a}_r + r \cos\theta \cos\phi\, \mathbf{a}_\theta - r \sin\phi\, \mathbf{a}_\phi$ C/m^2 at $P_C(r = 1.5, \theta = 30^\circ, \phi = 50^\circ)$.

Ans. −10.00; 9.06; 1.29

Finally, we can combine Eqs. (9) and (12) and form the relation between electric flux density and charge density:

$$\text{div}\,\mathbf{D} = \rho_v \tag{15}$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is aptly called the *point form of Gauss's law*. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point. Because the divergence may be expressed as the sum of three partial derivatives, Maxwell's first equation is also described as the differential-equation form of Gauss's law, and conversely, Gauss's law is recognized as the integral form of Maxwell's first equation.

As a specific illustration, let us consider the divergence of **D** in the region about a point charge Q located at the origin. We have the field

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

and use (14), the expression for divergence in spherical coordinates:

$$\text{div}\, \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

Because D_θ and D_ϕ are zero, we have

$$\text{div}\, \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0 \qquad (\text{if } r \neq 0)$$

Thus, $\rho_\nu = 0$ everywhere except at the origin, where it is infinite.

The divergence operation is not limited to electric flux density; it can be applied to any vector field. We will apply it to several other electromagnetic fields in the coming chapters.

D3.8. Determine an expression for the volume charge density associated with each $\mathbf{D}$ field: (a) $\mathbf{D} = \frac{4xy}{z}\mathbf{a}_x + \frac{2x^2}{z}\mathbf{a}_y - \frac{2x^2 y}{z^2}\mathbf{a}_z$; ($b$) $\mathbf{D} = z\sin\phi\,\mathbf{a}_\rho + z\cos\phi\,\mathbf{a}_\phi + \rho\sin\phi\,\mathbf{a}_z$; ($c$) $\mathbf{D} = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$.

Ans. $\frac{4y}{z^3}(x^2 + z^2)$; 0; 0.

3.6 THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something that may be dotted formally with $\mathbf{D}$ to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \qquad (16)$$

Similar *scalar operators* appear in several methods of solving differential equations where we often let D replace d/dx, D^2 replace d^2/dx^2, and so forth.[4] We agree on defining ∇ that it shall be treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars.

Consider $\nabla \cdot \mathbf{D}$, signifying

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

[4] This scalar operator D, which will not appear again, is not to be confused with the electric flux density.

We first consider the dot products of the unit vectors, discarding the six zero terms, and obtain the result that we recognize as the divergence of **D**:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div}(\mathbf{D})$$

The use of $\nabla \cdot \mathbf{D}$ is much more prevalent than that of div **D**, although both usages have their advantages. Writing $\nabla \cdot \mathbf{D}$ allows us to obtain simply and quickly the correct partial derivatives, but only in rectangular coordinates, as we will see. On the other hand, div **D** is an excellent reminder of the physical interpretation of divergence. We shall use the operator notation $\nabla \cdot \mathbf{D}$ from now on to indicate the divergence operation.

The vector operator ∇ is used not only with divergence, but also with several other very important operations that appear later. One of these is ∇u, where u is any scalar field, and leads to

$$\nabla u = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) u = \frac{\partial u}{\partial x}\mathbf{a}_x + \frac{\partial u}{\partial y}\mathbf{a}_y + \frac{\partial u}{\partial z}\mathbf{a}_z$$

The ∇ operator does not have a specific form in other coordinate systems. If we are considering **D** in cylindrical coordinates, then $\nabla \cdot \mathbf{D}$ still indicates the divergence of **D**, or

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

where this expression has been taken from Section 3.5. We have no form for ∇ itself to help us obtain this sum of partial derivatives. This means that ∇u, as yet unnamed but easily written in rectangular coordinates, cannot be expressed by us at this time in cylindrical coordinates. Such an expression will be obtained when ∇u is defined in Chapter 4.

We close our discussion of divergence by presenting a theorem that will be needed several times in later chapters, the *divergence theorem*. This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv$$

The first and last expressions constitute the divergence theorem,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv \qquad (17)$$

which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

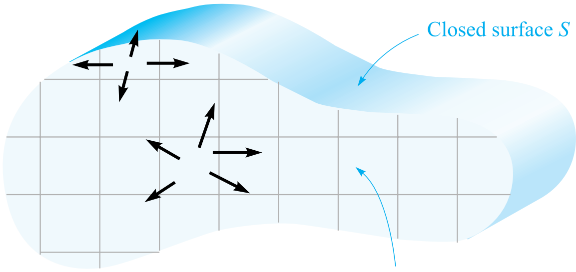

Figure 3.7 The divergence theorem states that the total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Again, we emphasize that the divergence theorem is true for any vector field, although we have obtained it specifically for the electric flux density **D**, and we will have occasion later to apply it to several different fields. Its benefits derive from the fact that it relates a triple integration *throughout some volume* to a double integration *over the surface* of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by inspecting the surface than by calculating the velocity at every internal point.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Figure 3.7, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell *enters*, or *converges* on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

EXAMPLE 3.5

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that **D** is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$
$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3 \int_0^2 (D_x)_{x=0} dy\, dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz$$

$$- \int_0^3 \int_0^1 (D_y)_{y=0} dx\, dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx\, dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz = 12$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y\, dx\, dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz = 12$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

D3.9. Given the field $\mathbf{D} = 6\rho \sin \frac{1}{2}\phi\, \mathbf{a}_\rho + 1.5\rho \cos \frac{1}{2}\phi\, \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region bounded by $\rho = 2$, $\phi = 0$, $\phi = \pi$, $z = 0$, and $z = 5$.

Ans. 225; 225

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1. Kraus, J. D., and D. A. Fleisch. *Electromagnetics*. 5th ed. New York: McGraw-Hill, 1999. The static electric field in free space is introduced in Chapter 2.
2. Plonsey, R., and R. E. Collin. *Principles and Applications of Electromagnetic Fields*. New York: McGraw-Hill, 1961. The level of this text is somewhat higher than the one we are reading now, but it is an excellent text to read next. Gauss's law appears in the second chapter.
3. Plonus, M. A. *Applied Electromagnetics*. New York: McGraw-Hill, 1978. This book contains rather detailed descriptions of many practical devices that illustrate electromagnetic applications. For example, see the discussion of xerography on pp. 95–98 as an electrostatics application.
4. Skilling, H. H. *Fundamentals of Electric Waves*. 2d ed. New York: John Wiley & Sons, 1948. The operations of vector calculus are well illustrated. Divergence is discussed on pp. 22 and 38. Chapter 1 is interesting reading.

5. Thomas, G. B., Jr., and R. L. Finney. (see Suggested References for Chapter 1). The divergence theorem is developed and illustrated from several different points of view on pp. 976–980.

CHAPTER 3 PROBLEMS

WW Quizzes

3.1 Suppose that the Faraday concentric sphere experiment is performed in free space using a central charge at the origin, Q_1, and with hemispheres of radius a. A second charge Q_2 (this time a point charge) is located at distance R from Q_1, where $R >> a$. (a) What is the force on the point charge before the hemispheres are assembled around Q_1? (b) What is the force on the point charge after the hemispheres are assembled but before they are discharged? (c) What is the force on the point charge after the hemispheres are assembled and after they are discharged? (d) Qualitatively, describe what happens as Q_2 is moved toward the sphere assembly to the extent that the condition $R >> a$ is no longer valid.

3.2 An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2).

3.3 The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_S = 5e^{-20|z|}$ nC/m^2. (a) What is the total amount of charge present? (b) How much electric flux leaves the surface $\rho = 8$ cm, 1 cm $< z <$ 5 cm, $30^\circ < \phi < 90^\circ$?

3.4 An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin.

3.5 Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2 + z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ nC/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m.

3.6 In free space, a volume charge of constant density $\rho_\nu = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find **D** and **E** everywhere.

3.7 Volume charge density is located in free space as $\rho_\nu = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_\nu = 0$ elsewhere. (a) Find the total charge enclosed by the spherical surface $r = 1$ mm. (b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm.

3.8 Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r.

3.9 A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\text{ mm} < r < 10\text{ mm}$. Let $\rho_v = 0$ for $0 < r < 8$ mm. (*a*) Find the total charge inside the spherical surface $r = 10$ mm. (*b*) Find D_r at $r = 10$ mm. (*c*) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm.

3.10 An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, **E**, both inside and outside the cylinder.

3.11 In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for $1\text{ mm} < \rho < 1.5$ mm, and $\rho_v = 0$ for $\rho > 1.5$ mm. Find **D** everywhere.

3.12 The sun radiates a total power of about 3.86×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation, (*a*) what power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W? (*b*) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

3.13 Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{S0}, respectively. (*a*) Find **D** at $r = 1$, 3, and 5 m. (*b*) Determine ρ_{S0} such that $\mathbf{D} = 0$ at $r = 7$ m.

3.14 A certain light-emitting diode (LED) is centered at the origin with its surface in the xy plane. At far distances, the LED appears as a point, but the glowing surface geometry produces a far-field radiation pattern that follows a raised cosine law: that is, the optical power (flux) density in watts/m^2 is given in spherical coordinates by

$$\mathbf{P}_d = P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r \qquad \text{watts/m}^2$$

where θ is the angle measured with respect to the direction that is normal to the LED surface (in this case, the z axis), and r is the radial distance from the origin at which the power is detected. (*a*) In terms of P_0, find the total power in watts emitted in the upper half-space by the LED; (*b*) Find the cone angle, θ_1, within which half the total power is radiated, that is, within the range $0 < \theta < \theta_1$; (*c*) An optical detector, having a 1-mm^2 cross-sectional area, is positioned at $r = 1$ m and at $\theta = 45°$, such that it faces the LED. If one milliwatt is measured by the detector, what (to a very good estimate) is the value of P_0?

3.15 Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\,\mu\text{C/m}^3$ for $1 < \rho < 2$ mm. (*a*) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm. (*b*) Use Gauss's law to determine D_ρ at $\rho = \rho_1$. (*c*) Evaluate D_ρ at $\rho = 0.8$ mm, 1.6 mm, and 2.4 mm.

3.16 An electric flux density is given by $\mathbf{D} = D_0\,\mathbf{a}_\rho$, where D_0 is a given constant. (*a*) What charge density generates this field? (*b*) For the specified field, what total charge is contained within a cylinder of radius a and height b, where the cylinder axis is the z axis?

3.17 A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2 (a) Apply Gauss's law to find the total flux leaving the closed surface of the cube. (b) Evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube. (c) Estimate the total charge enclosed within the cube by using Eq. (8).

3.18 State whether the divergence of the following vector fields is positive, negative, or zero: (a) the thermal energy flow in J/(m^2 − s) at any point in a freezing ice cube; (b) the current density in A/m^2 in a bus bar carrying direct current; (c) the mass flow rate in kg/(m^2 − s) below the surface of water in a basin, in which the water is circulating clockwise as viewed from above.

3.19 A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Section 3.4 to estimate the net electric flux leaving the spherical surface.

3.20 A radial electric field distribution in free space is given in spherical coordinates as:

$$\mathbf{E}_1 = \frac{r\rho_0}{3\epsilon_0}\,\mathbf{a}_r \qquad (r \le a)$$

$$\mathbf{E}_2 = \frac{(2a^3 - r^3)\rho_0}{3\epsilon_0 r^2}\,\mathbf{a}_r \qquad (a \le r \le b)$$

$$\mathbf{E}_3 = \frac{(2a^3 - b^3)\rho_0}{3\epsilon_0 r^2}\,\mathbf{a}_r \qquad (r \ge b)$$

where ρ_0, a, and b are constants. (a) Determine the volume charge density in the entire region ($0 \le r \le \infty$) by the appropriate use of $\nabla \cdot \mathbf{D} = \rho_v$. ($b$) In terms of given parameters, find the total charge, Q, within a sphere of radius r where $r > b$.

3.21 Calculate $\nabla \cdot \mathbf{D}$ at the point specified if (a) $\mathbf{D} = (1/z^2)[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z]$ at $P(-2, 3, 5)$; (b) $\mathbf{D} = 5z^2\,\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$; (c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$.

3.22 (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface, $r = a, 0 < \theta < \pi/2, 0 < \phi < 2\pi$. ($b$) What simple observation would have saved a lot of work in part a? (c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane. (d) Repeat part c by using the divergence theorem and an appropriate volume integral.

3.23 (a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. (b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere.

3.24 In a region in free space, electric flux density is found to be

$$\mathbf{D} = \begin{cases} \rho_0(z+2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (-2d \le z \le 0) \\ -\rho_0(z-2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (0 \le z \le 2d) \end{cases}$$

Everywhere else, $\mathbf{D} = 0$. (*a*) Using $\nabla \cdot \mathbf{D} = \rho_v$, find the volume charge density as a function of position everywhere. (*b*) Determine the electric flux that passes through the surface defined by $z = 0,\ -a \le x \le a,\ -b \le y \le b$. (*c*) Determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b,\ -d \le z \le d$. (*d*) Determine the total charge contained within the region $-a \le x \le a,\ -b \le y \le b,\ 0 \le z \le 2d$.

3.25 Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as $\mathbf{D} = 5(r-3)^3\ \mathbf{a}_r$ C/m^2. (*a*) What is the volume charge density at $r = 4$? (*b*) What is the electric flux density at $r = 4$? (*c*) How much electric flux leaves the sphere $r = 4$? (*d*) How much charge is contained within the sphere $r = 4$?

3.26 If we have a perfect gas of mass density ρ_m kg/m^3, and we assign a velocity **U** m/s to each differential element, then the mass flow rate is $\rho_m \mathbf{U}$ kg/(m^2 − s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m \mathbf{U}) = -\partial \rho_m / \partial t$. (*a*) Explain in words the physical interpretation of this equation. (*b*) Show that $\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface S, and explain the physical significance of the equation.

3.27 Let $\mathbf{D} = 5.00 r^2 \mathbf{a}_r$ mC/m^2 for $r \le 0.08$ m and $\mathbf{D} = 0.205\ \mathbf{a}_r / r^2$ μC/m^2 for $r \ge 0.08$ m. (*a*) Find ρ_v for $r = 0.06$ m. (*b*) Find ρ_v for $r = 0.1$ m. (*c*) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m?

3.28 Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.

3.29 In the region of free space that includes the volume $2 < x, y, z < 3$, $\mathbf{D} = \frac{2}{z^2}(yz\ \mathbf{a}_x + xz\ \mathbf{a}_y - 2xy\ \mathbf{a}_z)$ C/m^2. (*a*) Evaluate the volume integral side of the divergence theorem for the volume defined here. (*b*) Evaluate the surface integral side for the corresponding closed surface.

3.30 (*a*) Use Maxwell's first equation, $\nabla \cdot \mathbf{D} = \rho_v$, to describe the variation of the electric field intensity with x in a region in which no charge density exists and in which a nonhomogeneous dielectric has a permittivity that increases exponentially with x. The field has an x component only; (*b*) repeat part (*a*), but with a radially directed electric field (spherical coordinates), in which again $\rho_v = 0$, but in which the permittivity *decreases* exponentially with r.

3.31 Given the flux density $\mathbf{D} = \frac{16}{r}\cos(2\theta)\ \mathbf{a}_\theta$ C/m^2, use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad.

CHAPTER 4

Energy and Potential

In Chapters 2 and 3 we became acquainted with Coulomb's law and its use in finding the electric field about several simple distributions of charge, and also with Gauss's law and its application in determining the field about some symmetrical charge arrangements. The use of Gauss's law was invariably easier for these highly symmetrical distributions because the problem of integration always disappeared when the proper closed surface was chosen.

However, if we had attempted to find a slightly more complicated field, such as that of two unlike point charges separated by a small distance, we would have found it impossible to choose a suitable gaussian surface and obtain an answer. Coulomb's law, however, is more powerful and enables us to solve problems for which Gauss's law is not applicable. The application of Coulomb's law is laborious, detailed, and often quite complex, the reason for this being precisely the fact that the electric field intensity, a vector field, must be found directly from the charge distribution. Three different integrations are needed in general, one for each component, and the resolution of the vector into components usually adds to the complexity of the integrals.

Certainly it would be desirable if we could find some as yet undefined scalar function with a single integration and then determine the electric field from this scalar by some simple straightforward procedure, such as differentiation.

This scalar function does exist and is known as the *potential* or *potential field*. We shall find that it has a very real physical interpretation and is more familiar to most of us than is the electric field which it will be used to find.

We should expect, then, to be equipped soon with a third method of finding electric fields—a single scalar integration, although not always as simple as we might wish, followed by a pleasant differentiation.

4.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy or do work. If we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative; we do not do the work, the field does.

Suppose we wish to move a charge Q a distance $d\mathbf{L}$ in an electric field $\mathbf{E}$. The force on Q arising from the electric field is

$$\mathbf{F}_E = Q\mathbf{E} \tag{1}$$

where the subscript reminds us that this force arises from the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L =$ a unit vector in the direction of $d\mathbf{L}$.

The force that we must apply is equal and opposite to the force associated with the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

and the expenditure of energy is the product of the force and distance. That is, the differential work done by an external source moving charge Q is $dW = -Q\mathbf{E} \cdot \mathbf{a}_L dL$, or

$$dW = -Q\mathbf{E} \cdot d\mathbf{L} \tag{2}$$

where we have replaced $\mathbf{a}_L dL$ by the simpler expression $d\mathbf{L}$.

This differential amount of work required may be zero under several conditions determined easily from Eq. (2). There are the trivial conditions for which $\mathbf{E}$, Q, or $d\mathbf{L}$ is zero, and a much more important case in which $\mathbf{E}$ and $d\mathbf{L}$ are perpendicular. Here the charge is moved always in a direction at right angles to the electric field. We can draw on a good analogy between the electric field and the gravitational field, where, again, energy must be expended to move against the field. Sliding a mass around with constant velocity on a frictionless surface is an effortless process if the mass is moved along a constant elevation contour; positive or negative work must be done in moving it to a higher or lower elevation, respectively.

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating,

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \tag{3}$$

where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

This definite integral is basic to field theory, and we shall devote the following section to its interpretation and evaluation.

D4.1. Given the electric field $\mathbf{E} = \frac{1}{z^2}(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z)$ V/m, find the differential amount of work done in moving a 6-nC charge a distance of 2 µm, starting at $P(2, -2, 3)$ and proceeding in the direction $\mathbf{a}_L =$ (*a*) $-\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z$; (*b*) $\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z$; (*c*) $\frac{3}{7}\mathbf{a}_x + \frac{6}{7}\mathbf{a}_y$.

Ans. −149.3 fJ; 149.3 fJ; 0

4.2 THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q from one position to another, Eq. (3), is an example of a line integral, which in vector-analysis notation always takes the form of the integral along some prescribed path of the dot product of a vector field and a differential vector path length $d\mathbf{L}$. Without using vector analysis we should have to write

$$W = -Q \int_{\text{init}}^{\text{final}} E_L \, dL$$

where E_L = component of $\mathbf{E}$ along $d\mathbf{L}$.

A line integral is like many other integrals which appear in advanced analysis, including the surface integral appearing in Gauss's law, in that it is essentially descriptive. We like to look at it much more than we like to work it out. It tells us to choose a path, break it up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, and then add the results for all the segments. This is a summation, of course, and the integral is obtained exactly only when the number of segments becomes infinite.

This procedure is indicated in Figure 4.1, where a path has been chosen from an initial position B to a final position[1] A and a *uniform electric field* is selected for simplicity. The path is divided into six segments, $\Delta\mathbf{L}_1, \Delta\mathbf{L}_2, \ldots, \Delta\mathbf{L}_6$, and the components of $\mathbf{E}$ along each segment are denoted by $E_{L1}, E_{L2}, \ldots, E_{L6}$. The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

[1] The final position is given the designation A to correspond with the convention for potential difference, as discussed in the following section.

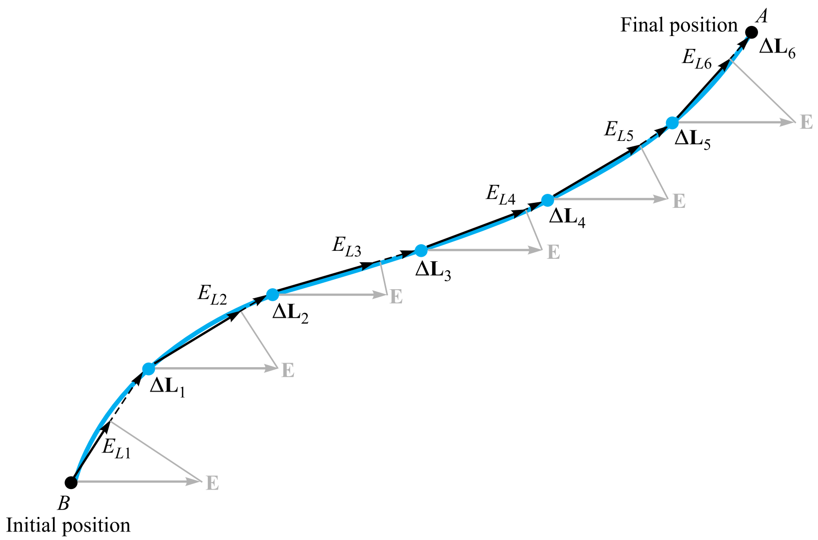

Figure 4.1 A graphical interpretation of a line integral in a uniform field. The line integral of **E** between points B and A is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

and because we have assumed a uniform field,

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$
$$W = -Q\mathbf{E}\cdot(\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)$$

What is this sum of vector segments in the preceding parentheses? Vectors add by the parallelogram law, and the sum is just the vector directed from the initial point B to the final point A, $\mathbf{L}_{BA}$. Therefore

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)} \tag{4}$$

Remembering the summation interpretation of the line integral, this result for the uniform field can be obtained rapidly now from the integral expression

$$W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L} \tag{5}$$

as applied to a uniform field

$$W = -Q\mathbf{E}\cdot\int_B^A d\mathbf{L}$$

where the last integral becomes $\mathbf{L}_{BA}$ and

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

For this special case of a uniform electric field intensity, we should note that the work involved in moving the charge depends only on Q, $\mathbf{E}$, and $\mathbf{L}_{BA}$, a vector drawn from the initial to the final point of the path chosen. It does not depend on the particular path we have selected along which to carry the charge. We may proceed from B to A on a straight line or via the Old Chisholm Trail; the answer is the same. We show in Section 4.5 that an identical statement may be made for any nonuniform (static) $\mathbf{E}$ field.

Let us use several examples to illustrate the mechanics of setting up the line integral appearing in Eq. (5).

EXAMPLE 4.1

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2C from $B(1, 0, 1)$ to $A(0.8, 0.6, 1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \quad z = 1$$

Solution. We use $W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$, where $\mathbf{E}$ is not necessarily constant. Working in rectangular coordinates, the differential path $d\mathbf{L}$ is $dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and the integral becomes

$$\begin{aligned} W &= -Q\int_B^A \mathbf{E}\cdot d\mathbf{L} \\ &= -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z)\cdot(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \\ &= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz \end{aligned}$$

where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variable of integration. Using the equation of the circular path (and selecting the sign of the radical which is correct for the quadrant involved), we have

$$\begin{aligned} W &= -2\int_1^{0.8} \sqrt{1-x^2}\,dx - 2\int_0^{0.6} \sqrt{1-y^2}\,dy - 0 \\ &= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_0^{0.6} \\ &= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) \\ &= -0.96\,\text{J} \end{aligned}$$

EXAMPLE 4.2

Again find the work required to carry 2C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. We start by determining the equations of the straight line. Any two of the following three equations for planes passing through the line are sufficient to define the line:

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first equation we have

$$y = -3(x - 1)$$

and from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$= 6\int_1^{0.8} (x-1)\,dx - 2\int_0^{0.6} \left(1 - \frac{y}{3}\right) dy$$

$$= -0.96 \text{ J}$$

This is the same answer we found using the circular path between the same two points, and it again demonstrates the statement (unproved) that the work done is independent of the path taken in any electrostatic field.

It should be noted that the equations of the straight line show that $dy = -3\,dx$ and $dx = -\frac{1}{3}\,dy$. These substitutions may be made in the first two integrals, along with a change in limits, and the answer may be obtained by evaluating the new integrals. This method is often simpler if the integrand is a function of only one variable.

Note that the expressions for $d\mathbf{L}$ in our three coordinate systems use the differential lengths obtained in Chapter 1 (rectangular in Section 1.3, cylindrical in Section 1.8, and spherical in Section 1.9):

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \quad \text{(rectangular)} \tag{6}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\mathbf{a}_\phi + dz\,\mathbf{a}_z \quad \text{(cylindrical)} \tag{7}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \quad \text{(spherical)} \tag{8}$$

The interrelationships among the several variables in each expression are determined from the specific equations for the path.

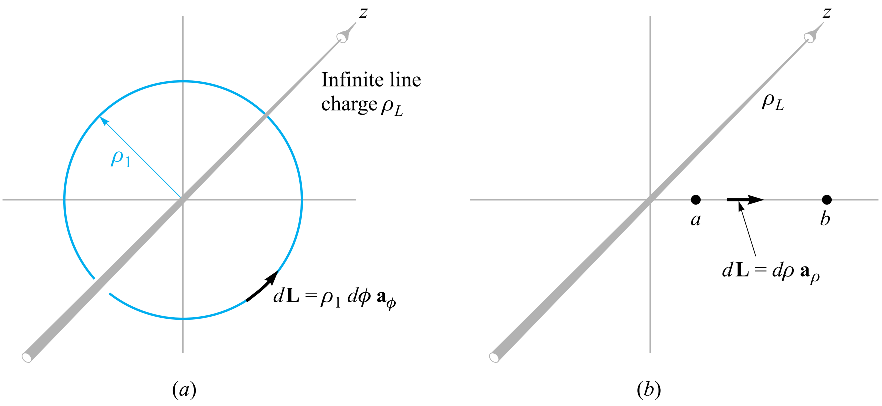

Figure 4.2 (*a*) A circular path and (*b*) a radial path along which a charge of Q is carried in the field of an infinite line charge. No work is expected in the former case.

As a final example illustrating the evaluation of the line integral, we investigate several paths that we might take near an infinite line charge. The field has been obtained several times and is entirely in the radial direction,

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

First we find the work done in carrying the positive charge Q about a circular path of radius ρ_b centered at the line charge, as illustrated in Figure 4.2a. Without lifting a pencil, we see that the work must be nil, for the path is always perpendicular to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it. For practice, however, we will set up the integral and obtain the answer.

The differential element $d\mathbf{L}$ is chosen in cylindrical coordinates, and the circular path selected demands that $d\rho$ and dz be zero, so $d\mathbf{L} = \rho_1\, d\phi\, \mathbf{a}_\phi$. The work is then

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\, d\phi\, \mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\, \mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

We will now carry the charge from $\rho = a$ to $\rho = b$ along a radial path (Figure 4.2b). Here $d\mathbf{L} = d\rho\, \mathbf{a}_\rho$ and

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho = -Q\int_a^b \frac{\rho_L}{2\pi\epsilon_0}\frac{d\rho}{\rho}$$

or

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Because b is larger than a, $\ln(b/a)$ is positive, and the work done is negative, indicating that the external source that is moving the charge receives energy.

One of the pitfalls in evaluating line integrals is a tendency to use too many minus signs when a charge is moved in the direction of a *decreasing* coordinate value. This is taken care of completely by the limits on the integral, and no misguided attempt should be made to change the sign of $d\mathbf{L}$. Suppose we carry Q from b to a (Figure 4.2b). We still have $d\mathbf{L} = d\rho\,\mathbf{a}_\rho$ and show the different direction by recognizing $\rho = b$ as the initial point and $\rho = a$ as the final point,

$$W = -Q \int_b^a \frac{\rho_L}{2\pi\epsilon_0} \frac{d\rho}{\rho} = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

This is the negative of the previous answer and is obviously correct.

D4.2. Calculate the work done in moving a 4-C charge from $B(1, 0, 0)$ to $A(0, 2, 0)$ along the path $y = 2 - 2x$, $z = 0$ in the field $\mathbf{E} =$ (a) $5\mathbf{a}_x$ V/m; (b) $5x\mathbf{a}_x$ V/m; (c) $5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m.

Ans. 20 J; 10 J; −30 J

D4.3. We will see later that a time-varying $\mathbf{E}$ field need not be conservative. (If it is not conservative, the work expressed by Eq. (3) may be a function of the path used.) Let $\mathbf{E} = y\mathbf{a}_x$ V/m at a certain instant of time, and calculate the work required to move a 3-C charge from (1, 3, 5) to (2, 0, 3) along the straight-line segments joining: (a) (1, 3, 5) to (2, 3, 5) to (2, 0, 5) to (2, 0, 3); (b) (1, 3, 5) to (1, 3, 3) to (1, 0, 3) to (2, 0, 3).

Ans. −9 J; 0

4.3 DEFINITION OF POTENTIAL DIFFERENCE AND POTENTIAL

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We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field $\mathbf{E}$, "Potential difference and work."

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

In much the same way as we defined the electric field intensity as the force on a *unit* test charge, we now define *potential difference V* as the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \tag{9}$$

We have to agree on the direction of movement, and we do this by stating that V_{AB} signifies the potential difference between points A and B and is the work done in moving the unit charge from B (last named) to A (first named). Thus, in determining V_{AB}, B is the initial point and A is the final point. The reason for this somewhat peculiar definition will become clearer shortly, when it is seen that the initial point B is often taken at infinity, whereas the final point A represents the fixed position of the charge; point A is thus inherently more significant.

Potential difference is measured in joules per coulomb, for which the *volt* is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} \text{ V} \tag{10}$$

and V_{AB} is positive if work is done in carrying the positive charge from B to A.

From the line-charge example of Section 4.2 we found that the work done in taking a charge Q from $\rho = b$ to $\rho = a$ was

$$W = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Thus, the potential difference between points at $\rho = a$ and $\rho = b$ is

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a} \tag{11}$$

We can try out this definition by finding the potential difference between points A and B at radial distances r_A and r_B from a point charge Q. Choosing an origin at Q,

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

and

$$d\mathbf{L} = dr\, \mathbf{a}_r$$

we have

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\, dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \tag{12}$$

If $r_B > r_A$, the potential difference V_{AB} is positive, indicating that energy is expended by the external source in bringing the positive charge from r_B to r_A. This agrees with the physical picture showing the two like charges repelling each other.

It is often convenient to speak of the *potential*, or *absolute potential*, of a point, rather than the potential difference between two points, but this means only that we agree to measure every potential difference with respect to a specified reference point that we consider to have zero potential. Common agreement must be reached on the zero reference before a statement of the potential has any significance. A person having one hand on the deflection plates of a cathode-ray tube that are "at a potential of 50 V" and the other hand on the cathode terminal would probably be too shaken up to understand that the cathode is not the zero reference, but that all potentials in that circuit are customarily measured with respect to the metallic shield about the tube. The cathode may be several thousands of volts negative with respect to the shield.

Perhaps the most universal zero reference point in experimental or physical potential measurements is "ground," by which we mean the potential of the surface region of the earth itself. Theoretically, we usually represent this surface by an infinite plane at zero potential, although some large-scale problems, such as those involving propagation across the Atlantic Ocean, require a spherical surface at zero potential.

Another widely used reference "point" is infinity. This usually appears in theoretical problems approximating a physical situation in which the earth is relatively far removed from the region in which we are interested, such as the static field near the wing tip of an airplane that has acquired a charge in flying through a thunderhead, or the field inside an atom. Working with the *gravitational* potential field on earth, the zero reference is normally taken at sea level; for an interplanetary mission, however, the zero reference is more conveniently selected at infinity.

A cylindrical surface of some definite radius may occasionally be used as a zero reference when cylindrical symmetry is present and infinity proves inconvenient. In a coaxial cable the outer conductor is selected as the zero reference for potential. And, of course, there are numerous special problems, such as those for which a two-sheeted hyperboloid or an oblate spheroid must be selected as the zero-potential reference, but these need not concern us immediately.

If the potential at point A is V_A and that at B is V_B, then

$$\boxed{V_{AB} = V_A - V_B} \tag{13}$$

where we necessarily agree that V_A and V_B shall have the same zero reference point.

D4.4. An electric field is expressed in rectangular coordinates by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find: (*a*) V_{MN} if points M and N are specified by $M(2, 6, -1)$ and $N(-3, -3, 2)$; (*b*) V_M if $V = 0$ at $Q(4, -2, -35)$; (*c*) V_N if $V = 2$ at $P(1, 2, -4)$.

Ans. -139.0 V; -120.0 V; 19.0 V

4.4 THE POTENTIAL FIELD OF A POINT CHARGE

In Section 4.3 we found an expression Eq. (12) for the potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin. How might we conveniently define a zero reference for potential? The simplest possibility is to let $V = 0$ at infinity. If we let the point at $r = r_B$ recede to infinity, the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

or, as there is no reason to identify this point with the A subscript,

$$V = \frac{Q}{4\pi\epsilon_0 r} \tag{14}$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference. Returning to a physical interpretation, we may say that $Q/4\pi\epsilon_0 r$ joules of work must be done in carrying a unit charge from infinity to any point r meters from the charge Q.

A convenient method to express the potential without selecting a specific zero reference entails identifying r_A as r once again and letting $Q/4\pi\epsilon_0 r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1 \tag{15}$$

and C_1 may be selected so that $V = 0$ at any desired value of r. We could also select the zero reference indirectly by electing to let V be V_0 at $r = r_0$.

It should be noted that the *potential difference* between two points is not a function of C_1.

Equations (14) and (15) represent the potential field of a point charge. The potential is a scalar field and does not involve any unit vectors.

We now define an *equipotential surface* as a surface composed of all those points having the same value of potential. All field lines would be perpendicular to such a surface at the points where they intersect it. Therefore, no work is involved in moving a unit charge around on an equipotential surface. The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

WW Illustrations

An inspection of the form of the potential field of a point charge shows that it is an inverse-distance field, whereas the electric field intensity was found to be an inverse-square-law function. A similar result occurs for the gravitational force field of a point mass (inverse-square law) and the gravitational potential field (inverse distance). The gravitational force exerted by the earth on an object one million miles from it is four times that exerted on the same object two million miles away. The kinetic energy given to a freely falling object starting from the end of the universe with zero velocity, however, is only twice as much at one million miles as it is at two million miles.

D4.5. A 15-nC point charge is at the origin in free space. Calculate V_1 if point P_1 is located at $P_1(-2, 3, -1)$ and (*a*) $V = 0$ at (6, 5, 4); (*b*) $V = 0$ at infinity; (*c*) $V = 5$ V at (2, 0, 4).

Ans. 20.67 V; 36.0 V; 10.89 V

4.5 THE POTENTIAL FIELD OF A SYSTEM OF CHARGES: CONSERVATIVE PROPERTY

The potential at a point has been defined as the work done in bringing a unit positive charge from the zero reference to the point, and we have suspected that this work, and hence the potential, is independent of the path taken. If it were not, potential would not be a very useful concept.

Let us now prove our assertion. We do so by beginning with the potential field of the single point charge for which we showed, in Section 4.4, the independence with regard to the path, noting that the field is linear with respect to charge so that superposition is applicable. It will then follow that the potential of a system of charges has a value at any point which is independent of the path taken in carrying the test charge to that point.

Thus the potential field of a single point charge, which we shall identify as Q_1 and locate at $\mathbf{r}_1$, involves only the distance $|\mathbf{r} - \mathbf{r}_1|$ from Q_1 to the point at $\mathbf{r}$ where we are establishing the value of the potential. For a zero reference at infinity, we have

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$

The potential arising from two charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is a function only of $|\mathbf{r} - \mathbf{r}_1|$ and $|\mathbf{r} - \mathbf{r}_2|$, the distances from Q_1 and Q_2 to the field point, respectively.

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}$$

Continuing to add charges, we find that the potential arising from n point charges is

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_m|} \tag{16}$$

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_\nu \Delta\nu$, then

$$V(\mathbf{r}) = \frac{\rho_\nu(\mathbf{r}_1)\Delta\nu_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{\rho_\nu(\mathbf{r}_2)\Delta\nu_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{\rho_\nu(\mathbf{r}_n)\Delta\nu_n}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_n|}$$

As we allow the number of elements to become infinite, we obtain the integral expression

$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|} \tag{17}$$

We have come quite a distance from the potential field of the single point charge, and it might be helpful to examine Eq. (17) and refresh ourselves as to the meaning of each term. The potential $V(\mathbf{r})$ is determined with respect to a zero reference potential at infinity and is an exact measure of the work done in bringing a unit charge from infinity to the field point at $\mathbf{r}$ where we are finding the potential. The volume charge density $\rho_v(\mathbf{r}')$ and differential volume element dv' combine to represent a differential amount of charge $\rho_v(\mathbf{r}')\,dv'$ located at $\mathbf{r}'$. The distance $|\mathbf{r}-\mathbf{r}'|$ is that distance from the source point to the field point. The integral is a multiple (volume) integral.

If the charge distribution takes the form of a line charge or a surface charge, the integration is along the line or over the surface:

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\,dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\,dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{19}$$

The most general expression for potential is obtained by combining Eqs. (16)–(19).

These integral expressions for potential in terms of the charge distribution should be compared with similar expressions for the electric field intensity, such as Eq. (15) in Section 2.3:

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

The potential again is inverse distance, and the electric field intensity, inverse-square law. The latter, of course, is also a vector field.

EXAMPLE 4.3

To illustrate the use of one of these potential integrals, we will find V on the z axis for a uniform line charge ρ_L in the form of a ring, $\rho = a$, in the $z = 0$ plane, as shown in Figure 4.3.

Solution. Working with Eq. (18), we have $dL' = a\,d\phi'$, $\mathbf{r} = z\mathbf{a}_z$, $\mathbf{r}' = a\mathbf{a}_\rho$, $|\mathbf{r}-\mathbf{r}'| = \sqrt{a^2+z^2}$, and

$$V = \int_0^{2\pi} \frac{\rho_L a\,d\phi'}{4\pi\epsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2+z^2}}$$

For a zero reference at infinity, then:

1. The potential arising from a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.
2. The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.
3. The potential arising from a number of point charges or any continuous charge distribution may therefore be found by carrying a unit charge from infinity to the point in question along any path we choose.

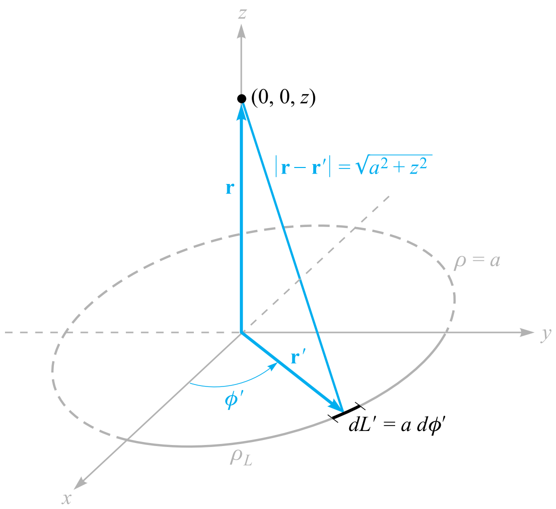

Figure 4.3 The potential field of a ring of uniform line charge density is easily obtained from $V = \int \rho_L(\mathbf{r}')\,dL'/(4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|)$.

In other words, the expression for potential (zero reference at infinity),

$$V_A = -\int_{\infty}^{A} \mathbf{E}\cdot d\mathbf{L}$$

or potential difference,

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \mathbf{E}\cdot d\mathbf{L}$$

is not dependent on the path chosen for the line integral, regardless of the source of the **E** field.

This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any *closed path*, or

$$\oint \mathbf{E}\cdot d\mathbf{L} = 0 \qquad (20)$$

A small circle is placed on the integral sign to indicate the closed nature of the path. This symbol also appeared in the formulation of Gauss's law, where a closed *surface* integral was used.

Equation (20) is true for *static* fields, but we will see in Chapter 9 that Faraday demonstrated it was incomplete when time-varying magnetic fields were present. One of Maxwell's greatest contributions to electromagnetic theory was in showing that a time-varying electric field produces a magnetic field, and therefore we should expect to find later that Eq. (20) is not correct when either **E** or the magnetic field varies with time.

Restricting our attention to the static case where **E** does not change with time, consider the dc circuit shown in Figure 4.4. Two points, A and B, are marked, and

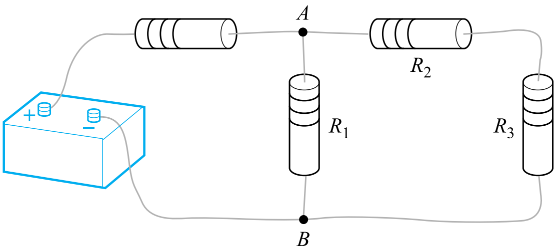

Figure 4.4 A simple dc-circuit problem that must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

(20) states that no work is involved in carrying a unit charge from A through R_2 and R_3 to B and back to A through R_1, or that the sum of the potential differences around any closed path is zero.

Equation (20) is therefore just a more general form of Kirchhoff's circuital law for voltages, more general in that we can apply it to any region where an electric field exists and we are not restricted to a conventional circuit composed of wires, resistances, and batteries. Equation (20) must be amended before we can apply it to time-varying fields.

Any field that satisfies an equation of the form of Eq. (20), (i.e., where the closed line integral of the field is zero) is said to be a *conservative field*. The name arises from the fact that no work is done (or that energy is *conserved*) around a closed path. The gravitational field is also conservative, for any energy expended in moving (raising) an object against the field is recovered exactly when the object is returned (lowered) to its original position. A nonconservative gravitational field could solve our energy problems forever.

Given a *nonconservative* field, it is of course possible that the line integral may be zero for certain closed paths. For example, consider the force field, $\mathbf{F} = \sin \pi\rho \, \mathbf{a}_\phi$. Around a circular path of radius $\rho = \rho_1$, we have $d\mathbf{L} = \rho \, d\phi \, \mathbf{a}_\phi$, and

$$\oint \mathbf{F} \cdot d\mathbf{L} = \int_0^{2\pi} \sin \pi\rho_1 \mathbf{a}_\phi \cdot \rho_1 d\phi \, \mathbf{a}_\phi = \int_0^{2\pi} \rho_1 \sin \pi\rho_1 \, d\phi$$
$$= 2\pi\rho_1 \sin \pi\rho_1$$

The integral is zero if $\rho_1 = 1, 2, 3, \ldots,$ etc., but it is not zero for other values of ρ_1, or for most other closed paths, and the given field is not conservative. A conservative field must yield a zero value for the line integral around every possible closed path.

D4.6. If we take the zero reference for potential at infinity, find the potential at (0, 0, 2) caused by this charge configuration in free space (*a*) 12 nC/m on the line $\rho = 2.5$ m, $z = 0$; (*b*) point charge of 18 nC at (1, 2, −1); (*c*) 12 nC/m on the line $y = 2.5$, $z = 0$, $-1.0 < x < 1.0$.

Ans. 529 V; 43.2 V; 66.3 V

4.6 POTENTIAL GRADIENT

We now have two methods of determining potential, one directly from the electric field intensity by means of a line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems, however, for as we will see later, neither the electric field intensity nor the charge distribution is very often known. Preliminary information is much more apt to consist of a description of two equipotential surfaces, such as the statement that we have two parallel conductors of circular cross section at potentials of 100 and −100 V. Perhaps we wish to find the capacitance between the conductors, or the charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method of finding the electric field intensity from the potential.

We already have the general line-integral relationship between these quantities,

$$V = -\int \mathbf{E} \cdot d\mathbf{L} \tag{21}$$

but this is much easier to use in the reverse direction: given **E**, find V.

However, Eq. (21) may be applied to a very short element of length $\Delta\mathbf{L}$ along which **E** is essentially constant, leading to an incremental potential difference ΔV,

$$\Delta V \doteq -\mathbf{E} \cdot \Delta\mathbf{L} \tag{22}$$

Now consider a general region of space, as shown in Figure 4.5, in which **E** and V both change as we move from point to point. Equation (22) tells us to choose an incremental vector element of length $\Delta\mathbf{L} = \Delta L\,\mathbf{a}_L$ and multiply its magnitude by

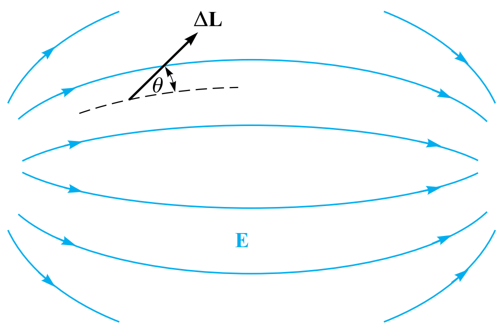

Figure 4.5 A vector incremental element of length ΔL is shown making an angle of θ with an E field, indicated by its streamlines. The sources of the field are not shown.

the component of **E** in the direction of $\mathbf{a}_L$ (one interpretation of the dot product) to obtain the small potential difference between the final and initial points of $\Delta\mathbf{L}$.

If we designate the angle between $\Delta\mathbf{L}$ and **E** as θ, then

$$\Delta V \doteq -E \Delta L \cos\theta$$

We now pass to the limit and consider the derivative dV/dL. To do this, we need to show that V may be interpreted as a *function* $V(x, y, z)$. So far, V is merely the result of the line integral (21). If we assume a specified starting point or zero reference and then let our end point be (x, y, z), we know that the result of the integration is a unique function of the end point (x, y, z) because **E** is a conservative field. Therefore V is a single-valued function $V(x, y, z)$. We may then pass to the limit and obtain

$$\frac{dV}{dL} = -E\cos\theta$$

In which direction should $\Delta\mathbf{L}$ be placed to obtain a maximum value of ΔV? Remember that **E** is a definite value at the point at which we are working and is independent of the direction of $\Delta\mathbf{L}$. The magnitude ΔL is also constant, and our variable is $\mathbf{a}_L$, the unit vector showing the direction of $\Delta\mathbf{L}$. It is obvious that the maximum positive increment of potential, ΔV_{max}, will occur when $\cos\theta$ is -1, or $\Delta\mathbf{L}$ points in the direction *opposite* to **E**. For this condition,

$$\left.\frac{dV}{dL}\right|_{\text{max}} = E$$

This little exercise shows us two characteristics of the relationship between **E** and V at any point:

1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.

2. This maximum value is obtained when the direction of the distance increment is opposite to **E** or, in other words, the direction of **E** is *opposite* to the direction in which the potential is *increasing* the most rapidly.

We now illustrate these relationships in terms of potential. Figure 4.6 is intended to show the information we have been given about some potential field. It does this by showing the equipotential surfaces (shown as lines in the two-dimensional sketch). We desire information about the electric field intensity at point P. Starting at P, we lay off a small incremental distance $\Delta\mathbf{L}$ in various directions, hunting for that direction in which the potential is changing (increasing) the most rapidly. From the sketch, this direction appears to be left and slightly upward. From our second characteristic above, the electric field intensity is therefore oppositely directed, or to the right and slightly downward at P. Its magnitude is given by dividing the small increase in potential by the small element of length.

It seems likely that the direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of *increasing* potential), and this is correct, for if $\Delta\mathbf{L}$ is directed along an equipotential, $\Delta V = 0$ by our

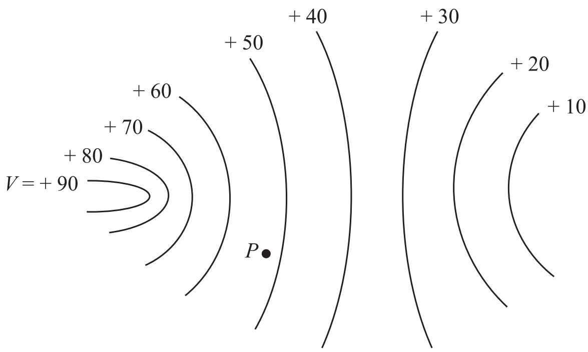

Figure 4.6 A potential field is shown by its equipotential surfaces. At any point the **E** field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.

definition of an equipotential surface. But then

$$\Delta V = -\mathbf{E} \cdot \Delta \mathbf{L} = 0$$

and as neither $\mathbf{E}$ nor $\Delta\mathbf{L}$ is zero, $\mathbf{E}$ must be perpendicular to this $\Delta\mathbf{L}$ or perpendicular to the equipotentials.

Because the potential field information is more likely to be determined first, let us describe the direction of $\Delta\mathbf{L}$, which leads to a maximum increase in potential mathematically in terms of the potential field rather than the electric field intensity. We do this by letting $\mathbf{a}_N$ be a unit vector normal to the equipotential surface and directed toward the higher potentials. The electric field intensity is then expressed in terms of the potential,

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N \tag{23}$$

which shows that the magnitude of $\mathbf{E}$ is given by the maximum space rate of change of V and the direction of $\mathbf{E}$ is *normal* to the equipotential surface (in the direction of *decreasing* potential).

Because $dV/dL|_{\max}$ occurs when $\Delta\mathbf{L}$ is in the direction of $\mathbf{a}_N$, we may remind ourselves of this fact by letting

$$\left.\frac{dV}{dL}\right|_{\max} = \frac{dV}{dN}$$

and

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N \tag{24}$$

Either Eq. (23) or Eq. (24) provides a physical interpretation of the process of finding the electric field intensity from the potential. Both are descriptive of a general procedure, and we do not intend to use them directly to obtain quantitative information.

This procedure leading from V to $\mathbf{E}$ is not unique to this pair of quantities, however, but has appeared as the relationship between a scalar and a vector field in hydraulics, thermodynamics, and magnetics, and indeed in almost every field to which vector analysis has been applied.

The operation on V by which $-\mathbf{E}$ is obtained is known as the *gradient*, and the gradient of a scalar field T is defined as

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\mathbf{a}_N \tag{25}$$

where $\mathbf{a}_N$ is a unit vector normal to the equipotential surfaces, and that normal is chosen, which points in the direction of increasing values of T.

Using this new term, we now may write the relationship between V and $\mathbf{E}$ as

$$\mathbf{E} = -\text{grad } V \tag{26}$$

Because we have shown that V is a unique function of x, y, and z, we may take its total differential

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

But we also have

$$dV = -\mathbf{E} \cdot d\mathbf{L} = -E_x\, dx - E_y\, dy - E_z\, dz$$

Because both expressions are true for any dx, dy, and dz, then

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially to yield

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right) \tag{27}$$

and comparing Eqs. (26) and (27) provides us with an expression which may be used to evaluate the gradient in rectangular coordinates,

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \tag{28}$$

The gradient of a scalar is a vector, and old quizzes show that the unit vectors that are often incorrectly added to the divergence expression appear to be those that were incorrectly removed from the gradient. Once the physical interpretation of the gradient, expressed by Eq. (25), is grasped as showing the maximum space rate of change of a scalar quantity and *the direction in which this maximum occurs*, the vector nature of the gradient should be self-evident.

The vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

may be used formally as an operator on a scalar, T, ∇T, producing

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{a}_x + \frac{\partial T}{\partial y}\mathbf{a}_y + \frac{\partial T}{\partial z}\mathbf{a}_z$$

from which we see that

$$\boxed{\nabla T = \text{grad}\, T}$$

This allows us to use a very compact expression to relate **E** and V,

$$\boxed{\mathbf{E} = -\nabla V} \tag{29}$$

The gradient may be expressed in terms of partial derivatives in other coordinate systems through the application of its definition Eq. (25). These expressions are derived in Appendix A and repeated here for convenience when dealing with problems having cylindrical or spherical symmetry. They also appear inside the back cover.

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(rectangular)}} \tag{30}$$

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \quad \text{(cylindrical)}} \tag{31}$$

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \quad \text{(spherical)}} \tag{32}$$

Note that the denominator of each term has the form of one of the components of $d\mathbf{L}$ in that coordinate system, except that partial differentials replace ordinary differentials; for example, $r \sin\theta\, d\phi$ becomes $r \sin\theta\, \partial\phi$.

We now illustrate the gradient concept with an example.

EXAMPLE 4.4

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \text{ V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \text{ V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\begin{aligned}\mathbf{a}_{E,P} &= (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9\\ &= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z\end{aligned}$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\,\mathbf{a}_x - 17.71x^2\,\mathbf{a}_y + 44.3\,\mathbf{a}_z \text{ pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla\cdot\mathbf{D} = -35.4y \text{ pC/m}^3$$

At P, $\rho_v = -106.2$ pC/m^3.

D4.7. A portion of a two-dimensional ($E_z = 0$) potential field is shown in Figure 4.7. The grid lines are 1 mm apart in the actual field. Determine approximate values for $\mathbf{E}$ in rectangular coordinates at: (*a*) *a*; (*b*) *b*; (*c*) *c*.

Ans. $-1075\mathbf{a}_y$ V/m; $-600\mathbf{a}_x - 700\mathbf{a}_y$ V/m; $-500\mathbf{a}_x - 650\mathbf{a}_y$ V/m

D4.8. Given the potential field in cylindrical coordinates, $V = \dfrac{100}{z^2+1}\rho\cos\phi$V, and point P at $\rho = 3$ m, $\phi = 60°$, $z = 2$ m, find values at P for (*a*) V; (*b*) $\mathbf{E}$; (*c*) E; (*d*) dV/dN; (*e*) $\mathbf{a}_N$; (*f*) ρ_v in free space.

Ans. 30.0 V; $-10.00\mathbf{a}_\rho + 17.3\mathbf{a}_\phi + 24.0\mathbf{a}_z$ V/m; 31.2 V/m; 31.2 V/m; $0.32\mathbf{a}_\rho - 0.55\mathbf{a}_\phi - 0.77\mathbf{a}_z$; -234 pC/m^3

4.7 THE ELECTRIC DIPOLE

The dipole fields that we develop in this section are quite important because they form the basis for the behavior of dielectric materials in electric fields, as discussed in Chapter 6, as well as justifying the use of images, as described in Section 5.5 of Chapter 5. Moreover, this development will serve to illustrate the importance of the potential concept presented in this chapter.

An *electric dipole*, or simply a *dipole*, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance that is small compared to

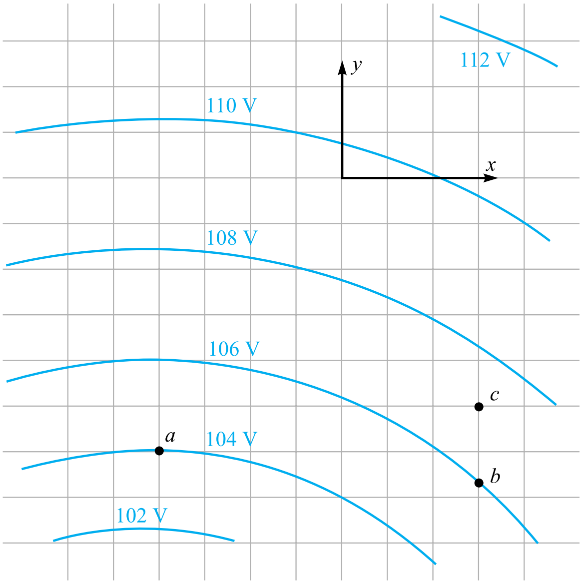

Figure 4.7 See Problem D4.7.

the distance to the point P at which we want to know the electric and potential fields. The dipole is shown in Figure 4.8a. The distant point P is described by the spherical coordinates r, θ, and $\phi = 90^\circ$, in view of the azimuthal symmetry. The positive and negative point charges have separation d and rectangular coordinates $(0, 0, \frac{1}{2}d)$ and $(0, 0, -\frac{1}{2}d)$, respectively.

So much for the geometry. What would we do next? Should we find the total electric field intensity by adding the known fields of each point charge? Would it be easier to find the total potential field first? In either case, having found one, we will find the other from it before calling the problem solved.

If we choose to find **E** first, we will have two components to keep track of in spherical coordinates (symmetry shows E_ϕ is zero), and then the only way to find V from **E** is by use of the line integral. This last step includes establishing a suitable zero reference for potential, since the line integral gives us only the potential difference between the two points at the ends of the integral path.

On the other hand, the determination of V first is a much simpler problem. This is because we find the potential as a function of position by simply adding the scalar potentials from the two charges. The position-dependent vector magnitude and direction of **E** are subsequently evaluated with relative ease by taking the negative gradient of V.

Choosing this simpler method, we let the distances from Q and $-Q$ to P be R_1 and R_2, respectively, and write the total potential as

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

CHAPTER 4 PROBLEMS

Quizzes

4.1 The value of **E** at $P(\rho = 2, \phi = 40^\circ, z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm: (*a*) in the direction of $\mathbf{a}_\rho$; (*b*) in the direction of $\mathbf{a}_\phi$; (*c*) in the direction of $\mathbf{a}_z$; (*d*) in the direction of **E**; (*e*) in the direction of $\mathbf{G} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$.

4.2 A positive point charge of magnitude q_1 lies at the origin. Derive an expression for the incremental work done in moving a second point charge q_2 through a distance dx from the starting position (x, y, z), in the direction of $-\mathbf{a}_x$.

4.3 If $\mathbf{E} = 120\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50-μC charge a distance of 2 mm from (*a*) $P(1, 2, 3)$ toward $Q(2, 1, 4)$; (*b*) $Q(2, 1, 4)$ toward $P(1, 2, 3)$.

4.4 An electric field in free space is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work done in moving a 1-μC charge through this field (*a*) from (1, 1, 1) to (0, 0, 0); (*b*) from $(\rho = 2, \phi = 0)$ to $(\rho = 2, \phi = 90^\circ)$; (*c*) from $(r = 10, \theta = \theta_0)$ to $(r = 10, \theta = \theta_0 + 180^\circ)$.

4.5 Compute the value of $\int_A^P \mathbf{G} \cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path (*a*) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$; (*b*) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$.

4.6 An electric field in free space is given as $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z$. Given $V(1, 1, 1) = 10$ V, determine $V(3, 3, 3)$.

4.7 Let $\mathbf{G} = 3xy^2\mathbf{a}_x + 2z\mathbf{a}_y$ Given an initial point $P(2, 1, 1)$ and a final point $Q(4, 3, 1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path (*a*) straight line: $y = x - 1$, $z = 1$; (*b*) parabola: $6y = x^2 + 2$, $z = 1$.

4.8 Given $\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$, (*a*) find the work involved in moving a unit positive charge on a circular arc, the circle centered at the origin, from $x = a$ to $x = y = a/\sqrt{2}$; (*b*) verify that the work done in moving the charge around the full circle from $x = a$ is zero.

4.9 A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space. (*a*) Find the absolute potential at $P(r = 1 \text{ cm}, \theta = 25^\circ, \phi = 50^\circ)$. (*b*) Find V_{AB}, given points $A(r = 2 \text{ cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3 \text{ cm}, \theta = 45^\circ, \phi = 90^\circ)$.

4.10 A sphere of radius a carries a surface charge density of ρ_{s0} C/m^2. (*a*) Find the absolute potential at the sphere surface. (*b*) A grounded conducting shell of radius b where $b > a$ is now positioned around the charged sphere. What is the potential at the inner sphere surface in this case?

4.11 Let a uniform surface charge density of 5 nC/m^2 be present at the $z = 0$ plane, a uniform line charge density of 8 nC/m be located at $x = 0, z = 4$, and a point charge of 2 μC be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$.

4.12 In spherical coordinates, $\mathbf{E} = 2r/(r^2 + a^2)^2\mathbf{a}_r$ V/m. Find the potential at any point, using the reference $(a) V = 0$ at infinity; (b) $V = 0$ at $r = 0$; $(c) V = 100$ V at $r = a$.

4.13 Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them?

4.14 Given the electric field $\mathbf{E} = (y + 1)\mathbf{a}_x + (x - 1)\mathbf{a}_y + 2\mathbf{a}_z$ find the potential difference between the points (a) $(2, -2, -1)$ and $(0, 0, 0)$; (b) $(3, 2, -1)$ and $(-2, -3, 4)$.

4.15 Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$.

4.16 A spherically symmetric charge distribution in free space (with $0 < r < \infty$) is known to have a potential function $V(r) = V_0a^2/r^2$, where V_0 and a are constants. (a) Find the electric field intensity. (b) Find the volume charge density. (c) Find the charge contained inside radius a. (d) Find the total energy stored in the charge (or equivalently, in its electric field).

4.17 Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm, respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at (a) $\rho = 5$ cm; (b) $\rho = 7$ cm.

4.18 Find the potential at the origin produced by a line charge $\rho_L = kx/(x^2 + a^2)$ extending along the x axis from $x = a$ to $+\infty$, where $a > 0$. Assume a zero reference at infinity.

4.19 The annular surface 1 cm $< \rho <$ 3 cm, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho$ nC/m^2. Find V at $P(0, 0, 2$ cm$)$ if $V = 0$ at infinity.

4.20 In a certain medium, the electric potential is given by

$$V(x) = \frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)$$

where ρ_0 and a are constants. (a) Find the electric field intensity, $\mathbf{E}$. (b) Find the potential difference between the points $x = d$ and $x = 0$. (c) If the medium permittivity is given by $\epsilon(x) = \epsilon_0 e^{ax}$, find the electric flux density, $\mathbf{D}$, and the volume charge density, ρ_v, in the region. (d) Find the stored energy in the region $(0 < x < d)$, $(0 < y < 1)$, $(0 < z < 1)$.

4.21 Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$ (a) V; (b) $|V|$; (c) $\mathbf{E}$; (d) $|\mathbf{E}|$; (e) $\mathbf{a}_N$; (f) $\mathbf{D}$.

4.22 A line charge of infinite length lies along the z axis and carries a uniform linear charge density of ρ_ℓ C/m. A perfectly conducting cylindrical shell, whose axis is the z axis, surrounds the line charge. The cylinder (of radius b), is at ground potential. Under these conditions, the potential function inside the cylinder ($\rho < b$) is given by

$$V(\rho) = k - \frac{\rho_\ell}{2\pi\epsilon_0}\ln(\rho)$$

where k is a constant. (a) Find k in terms of given or known parameters. (b) Find the electric field strength, $\mathbf{E}$, for $\rho < b$. (c) Find the electric field strength, $\mathbf{E}$, for $\rho > b$. (d) Find the stored energy in the electric field *per unit length* in the z direction within the volume defined by $\rho > a$, where $a < b$.

4.23 It is known that the potential is given as $V = 80\rho^{0.6}$ V. Assuming free space conditions, find. (a) $\mathbf{E}$; (b) the volume charge density at $\rho = 0.5$ m; (c) the total charge lying within the closed surface $\rho = 0.6$, $0 < z < 1$.

4.24 A certain spherically symmetric charge configuration in free space produces an electric field given in spherical coordinates by

$$\mathbf{E}(r) = \begin{cases} (\rho_0 r^2)/(100\epsilon_0)\,\mathbf{a}_r \text{ V/m} & (r \le 10) \\ (100\rho_0)/(\epsilon_0 r^2)\,\mathbf{a}_r \text{ V/m} & (r \ge 10) \end{cases}$$

where ρ_0 is a constant. (a) Find the charge density as a function of position. (b) Find the absolute potential as a function of position in the two regions, $r \le 10$ and $r \ge 10$. (c) Check your result of part b by using the gradient. (d) Find the stored energy in the charge by an integral of the form of Eq. (43). (e) Find the stored energy in the field by an integral of the form of Eq. (45).

4.25 Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho\sin\phi$V. ($a$) Find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space. (b) How much charge lies within the cylinder?

4.26 Let us assume that we have a very thin, square, imperfectly conducting plate 2 m on a side, located in the plane $z = 0$ with one corner at the origin such that it lies entirely within the first quadrant. The potential at any point in the plate is given as $V = -e^{-x}\sin y$. (a) An electron enters the plate at $x = 0$, $y = \pi/3$ with zero initial velocity; in what direction is its initial movement? (b) Because of collisions with the particles in the plate, the electron achieves a relatively low velocity and little acceleration (the work that the field does on it is converted largely into heat). The electron therefore moves approximately along a streamline. Where does it leave the plate and in what direction is it moving at the time?

4.27 Two point charges, 1 nC at (0, 0, 0.1) and −1 nC at (0, 0, −0.1), are in free space. (a) Calculate V at $P(0.3, 0, 0.4)$. (b) Calculate $|\mathbf{E}|$ at P. (c) Now treat the two charges as a dipole at the origin and find V at P.

4.28 Use the electric field intensity of the dipole [Section 4.7, Eq. (35)] to find the difference in potential between points at θ_a and θ_b, each point having the same r and ϕ coordinates. Under what conditions does the answer agree with Eq. (33), for the potential at θ_a?

4.29 A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$.

4.30 A dipole for which $\mathbf{p} = 10\epsilon_0\mathbf{a}_z$ C · m is located at the origin. What is the equation of the surface on which $E_z = 0$ but $\mathbf{E} \neq 0$?

4.31 A potential field in free space is expressed as $V = 20/(xyz)$ V. (*a*) Find the total energy stored within the cube $1 < x, y, z < 2$. (*b*) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube?

4.32 (*a*) Using Eq. (35), find the energy stored in the dipole field in the region $r > a$. (*b*) Why can we not let a approach zero as a limit?

4.33 A copper sphere of radius 4 cm carries a uniformly distributed total charge of 5 μC in free space. (*a*) Use Gauss's law to find $\mathbf{D}$ external to the sphere. (*b*) Calculate the total energy stored in the electrostatic field. (*c*) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere.

4.34 A sphere of radius a contains volume charge of uniform density ρ_0 C/m^3. Find the total stored energy by applying (*a*) Eq. (42); (*b*) Eq. (44).

4.35 Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side. (*a*) Find the total potential energy stored. (*b*) A fifth 0.8 nC charge is installed at the center of the square. Again find the total stored energy.

4.36 Surface charge of uniform density ρ_s lies on a spherical shell of radius b, centered at the origin in free space. (*a*) Find the absolute potential everywhere, with zero reference at infinity. (*b*) Find the stored energy in the sphere by considering the charge density and the potential in a two-dimensional version of Eq. (42). (*c*) Find the stored energy in the electric field and show that the results of parts (*b*) and (*c*) are identical.

CHAPTER 5

Conductors and Dielectrics

In this chapter, we apply the methods we have learned to some of the materials with which an engineer must work. In the first part of the chapter, we consider conducting materials by describing the parameters that relate current to an applied electric field. This leads to a general definition of Ohm's law. We then develop methods of evaluating resistances of conductors in a few simple geometric forms. Conditions that must be met at a conducting boundary are obtained next, and this knowledge leads to a discussion of the method of images. The properties of semiconductors are described to conclude the discussion of conducting media.

In the second part of the chapter, we consider insulating materials, or dielectrics. Such materials differ from conductors in that ideally, there is no free charge that can be transported within them to produce conduction current. Instead, all charge is confined to molecular or lattice sites by coulomb forces. An applied electric field has the effect of displacing the charges slightly, leading to the formation of ensembles of electric dipoles. The extent to which this occurs is measured by the relative permittivity, or dielectric constant. Polarization of the medium may modify the electric field, whose magnitude and direction may differ from the values it would have in a different medium or in free space. Boundary conditions for the fields at interfaces between dielectrics are developed to evaluate these differences.

It should be noted that most materials will possess both dielectric and conductive properties; that is, a material considered a dielectric may be slightly conductive, and a material that is mostly conductive may be slightly polarizable. These departures from the ideal cases lead to some interesting behavior, particularly as to the effects on electromagnetic wave propagation, as we will see later. ■

5.1 CURRENT AND CURRENT DENSITY

Electric charges in motion constitute a *current*. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I, and therefore

$$I = \frac{dQ}{dt} \tag{1}$$

Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons, as we will see shortly.

In field theory, we are usually interested in events occurring at a point rather than within a large region, and we find the concept of *current density,* measured in amperes per square meter (A/m^2), more useful. Current density is a vector[1] represented by $\mathbf{J}$.

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S$$

and in the case where the current density is not perpendicular to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{2}$$

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$, as shown in Figure 5.1*a*. To simplify the explanation, assume that the charge element is oriented with its edges parallel to the coordinate axes and that it has only an x component of velocity. In the time interval Δt, the element of charge has moved a distance Δx, as indicated in Figure 5.1*b*. We have therefore moved a charge $\Delta Q = \rho_v \, \Delta S \, \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt, and the resulting current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit with respect to time, we have

$$\Delta I = \rho_v \, \Delta S \, v_x$$

[1] Current is not a vector, for it is easy to visualize a problem in which a total current I in a conductor of nonuniform cross section (such as a sphere) may have a different direction at each point of a given cross section. Current in an exceedingly fine wire, or a *filamentary current*, is occasionally defined as a vector, but we usually prefer to be consistent and give the direction to the filament, or path, and not to the current.

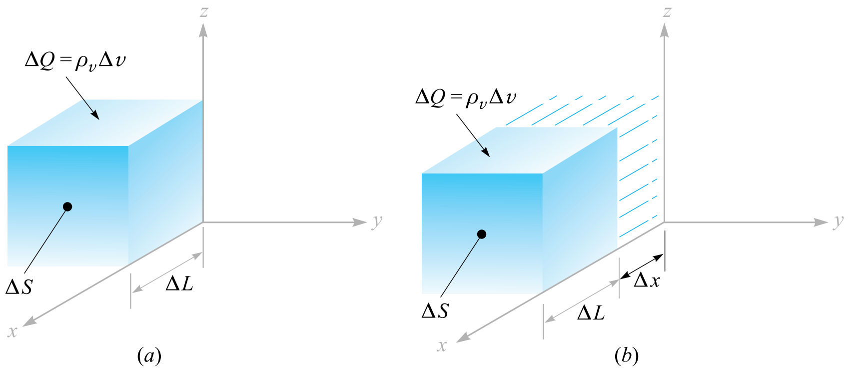

Figure 5.1 An increment of charge, $\Delta Q = \rho_v \Delta S \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

where v_x represents the x component of the velocity $\mathbf{v}$.[2] In terms of current density, we find

$$J_x = \rho_v \, v_x$$

and in general

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \tag{3}$$

This last result shows clearly that charge in motion constitutes a current. We call this type of current a *convection current,* and $\mathbf{J}$ or $\rho_v \mathbf{v}$ is the *convection current density*. Note that the convection current density is related linearly to charge density as well as to velocity. The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

D5.1. Given the vector current density $\mathbf{J} = 10\rho^2 z\mathbf{a}_\rho - 4\rho\cos^2\phi\,\mathbf{a}_\phi$ mA/m^2: (a) find the current density at $P(\rho = 3, \phi = 30°, z = 2)$; ($b$) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180\mathbf{a}_\rho - 9\mathbf{a}_\phi$ mA/m^2; 3.26 A

5.2 CONTINUITY OF CURRENT

The introduction of the concept of current is logically followed by a discussion of the conservation of charge and the continuity equation. The principle of conservation of charge states simply that charges can be neither created nor destroyed, although equal

[2] The lowercase v is used both for volume and velocity. Note, however, that velocity always appears as a vector $\mathbf{v}$, a component v_x, or a magnitude $|\mathbf{v}|$, whereas volume appears only in differential form as dv or Δv.

amounts of positive and negative charge may be *simultaneously* created, obtained by separation, or lost by recombination.

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow *into* one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an *outward-flowing* current.

Equation (4) is the integral form of the continuity equation; the differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v\, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t}\, dv$$

from which we have our point form of the continuity equation,

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t} \tag{5}$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

As a numerical example illustrating some of the concepts from the last two sections, let us consider a current density that is directed radially outward and decreases exponentially with time,

$$\mathbf{J} = \frac{1}{r} e^{-t} \mathbf{a}_r \text{ A/m}^2$$

Selecting an instant of time $t = 1$ s, we may calculate the total outward current at $r = 5$ m:

$$I = J_r S = \left(\tfrac{1}{5}e^{-1}\right)(4\pi 5^2) = 23.1 \text{ A}$$

At the same instant, but for a slightly larger radius, $r = 6$ m, we have

$$I = J_r S = \left(\tfrac{1}{6}e^{-1}\right)\left(4\pi 6^2\right) = 27.7 \text{ A}$$

Thus, the total current is larger at $r = 6$ than it is at $r = 5$.

To see why this happens, we need to look at the volume charge density and the velocity. We use the continuity equation first:

$$-\frac{\partial \rho_\nu}{\partial t} = \nabla \cdot \mathbf{J} = \nabla \cdot \left(\frac{1}{r}e^{-t}\mathbf{a}_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}e^{-t}\right) = \frac{1}{r^2}e^{-t}$$

We next seek the volume charge density by integrating with respect to t. Because ρ_ν is given by a partial derivative with respect to time, the "constant" of integration may be a function of r:

$$\rho_\nu = -\int \frac{1}{r^2}e^{-t}\,dt + K(r) = \frac{1}{r^2}e^{-t} + K(r)$$

If we assume that $\rho_\nu \rightarrow 0$ as $t \rightarrow \infty$, then $K(r) = 0$, and

$$\rho_\nu = \frac{1}{r^2}e^{-t} \text{ C/m}^3$$

We may now use $\mathbf{J} = \rho_\nu \mathbf{v}$ to find the velocity,

$$v_r = \frac{J_r}{\rho_\nu} = \frac{\frac{1}{r}e^{-t}}{\frac{1}{r^2}e^{-t}} = r \text{ m/s}$$

The velocity is greater at $r = 6$ than it is at $r = 5$, and we see that some (unspecified) force is accelerating the charge density in an outward direction.

In summary, we have a current density that is inversely proportional to r, a charge density that is inversely proportional to r^2, and a velocity and total current that are proportional to r. All quantities vary as e^{-t}.

D5.2. Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5}\mathbf{a}_z$ A/m^2 in the region $0 \leq \rho \leq 20\,\mu$m; for $\rho \geq 20\,\mu$m, $\mathbf{J} = 0$. (*a*) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction. (*b*) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_ν there. (*c*) If the volume charge density at $z = 0.15$ m is -2000 C/m^3, find the charge velocity there.

Ans. $-39.7\,\mu$A; -15.8 mC/m^3; 29.0 m/s